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ANALYTICAL COMPUTATION OF TWO INTEGRALS, APPEARING IN THE THEORY OF ELLIPTICAL ACCRETION DISCS. III. SOLVING OF THE FULL SET OF AUXILIARY INTEGRALS, CONTAINING LOGARITHMIC FUNCTIONS INTO THEIR INTEGRANDS

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Abstract

The present investigation encloses the started in the earlier papers [3] and [4] analytical evaluations of some kinds definite integrals. These solutions are necessary steps towards the revealing the mathematical structure of the dynamical equation, governing the properties of the stationary elliptical accretion discs, which apse lines of all orbits are in line with each other[5]. Though the considered here task, at first glance, may seem as a purely mathematical one, there are some restrictions of physical nature on the variables, entering as arguments into the integrals. In this paper we resolve analytically the following two definite integrals, including into their nominators (as a factor) the logarithmic function $\ln(1 + e\cos\varphi)$. *Concretely, we find in an explicit form the solutions of the integrals* $L_i(e,\dot{e}) \equiv$

$$\equiv \int_0^{2\pi} [\ln(1 + e\cos\varphi)](1 + e\cos\varphi)^{-1}[1 + (e - \dot{e})\cos\varphi]^{-i}\, d\varphi, \quad (i = 0,\dots, 3),$$

and $K_j(e,\dot{e}) \equiv \int_0^{2\pi} [\ln(1 + e\cos\varphi)]\, [1 + (e - \dot{e})\cos\varphi]^{-j}\, d\varphi, \quad (j = 1, \dots, 5).$

Here we have used the following notations: φ *is the azimuthal angle. The integration over* φ *from 0 to* 2π *means an averaging over the whole trajectory for each disc particle. Each such particle spirals inward to the center of the disc, moving on (quasi-) elliptical orbits with focal parameters* p. *These parameters* p *are allowed to vary for different elliptical orbits.In the our approach of computations, we treat* $e(u)$ *and* $\dot{e}(u)$ *as independent variables. The physically imposed restrictions (which, to some extend, lead to simplifications of the problems) are* $|e(u)| < 1$, $|\dot{e}(u)| < 1$ *and* $|e(u) - \dot{e}(u)| < 1$ *for all admitted values of* u. *That is to say, between the innermost and outermost orbits of the disc. Consequently, the established in this paper analytical solutions for the integrals* $L_i(e,\dot{e})$,

($\boldsymbol{i}$ = 0,…, 3) and $\boldsymbol{K_j(e,\dot{e})}$, ($\boldsymbol{j}$ = 1, …, 5), are, probably, not the most general ones, even in the domain of the real analysis. But nevertheless, they are sufficient for our aim to simplify the dynamical equation.

1. Introduction

Recent investigations ([1] – [4] and the references therein) have shown that it is possible to simplify to some extend the dynamical equation of the *stationary* elliptical accretion discs, having apse lines of the particle orbits in line with each other. This class of models, and, correspondingly, the accompanying their description dynamical equation, was developed by Lyubarskij et al. [5]. The simplifications, adopted for such models, allow to write the later equation as a second order ordinary differential equation. According to the theory of the ordinary differential equations, its solution exists and is unique. Of course, under given physically motivated suitable initial and boundary conditions. We underline that our up to now, and also forthcoming investigations are dealing only with the subclass of the *stationary* accretion flows. That is to say, for simplicity restrictions, we select only this part of the models, considered by Lyubarskij et al. [5], which does not concern the evolution of discs with the time. We shall not discuss here how the knowledge of the solutions for the *stationary* discs may hint the finding of the solutions of the dynamical equation in the *non-stationary* case. The property, that the *space* structure of the elliptical accretion disc (*having*, *as we just stressed above*, *characteristics which do not evolve with the time*) is described mathematically by an ordinary differential equation, stimulates our intention to try to solve it analytically. Or simplify it, by means of analytical transformations, to a form, which reveals in a more clear way its physical and mathematical interpretation. This situation is, evidently, much more easy for an analytical treatment, than the case when the particle orbits of the elliptical discs do not share a common longitude of periastron. Then the dynamical equation, governing the structure of the accretion flow, is, generally speaking, a *partial* differential equation [6]. It is known that the partial differential equations, in contrast to the ordinary such, do not posses guarantees that their solutions are unique, if the later exist at all!

During the process of finding of linear relations between the terms, entering into the dynamical equation of the disc, we strike with the necessity to compute the following two kinds of integrals:

(1) $\mathbf{L_i}(e,\dot{e}) \equiv \int_0^{2\pi} [ln(1 + ecos\varphi)](1 + ecos\varphi)^{-1}[1 + (e - \dot{e})cos\varphi]^{-\mathrm{i}}\, d\varphi$; $\mathbf{i} = 0, \ldots, 3$,

(2) $\mathbf{K_j}(e,\dot{e}) \equiv \int_0^{2\pi} [ln(1 + ecos\varphi)][1 + (e - \dot{e})cos\varphi]^{-\mathrm{j}}\, d\varphi$; $\mathbf{j} = 1, \ldots, 5$.

In an earlier paper [4], we have already evaluated analytically the "initial" integrals $\mathbf{L_0}(e)$, $\mathbf{K_0}(e)$ and $\mathbf{K_1}(e,\dot{e})$, considered as starting points into the recurrence relations, which enable us to solve the integrals with $\mathbf{i} = 1, 2, 3$ and $\mathbf{j} = 2, \ldots, 5$. We rewrite here these solutions:

(3) $\mathbf{L_0}(e) = -2\pi(1 - e^2)^{-1/2} ln\{[1 + (1 - e^2)^{1/2}][2(1 - e^2)]^{-1}\}$; ([4], equality (53)),

(4) $\mathbf{K_0}(e) = 2\pi\, ln\{[1 + (1 - e^2)^{1/2}]/2\}$; ([4], equality (54)),

(5) $\mathbf{K_1}(e,\dot{e}) = 2\pi[1 - (e - \dot{e})^2]^{-1/2} ln\boldsymbol{\{}\{2 - 3e^2 + e^4 + 3e\dot{e} - 2e^3\dot{e} - \dot{e}^2 + e^2\dot{e}^2 +$
$+ (-2 + 2e^2 - 3e\dot{e} + \dot{e}^2)(1 - e^2)^{1/2} + (-2 + 2e^2 - e\dot{e})[1 - (e - \dot{e})^2]^{1/2} +$
$+ (2 - e^2 + e\dot{e})(1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2}\}(e - \dot{e})^{-2}[1 - (1 - e^2)^{1/2}]^{-1}\boldsymbol{\}}$; ([4], equality (116)).

For further use, it is helpful also to adduce the definitions of the following three integrals $\mathbf{A_i}(e,\dot{e})$, ($\mathbf{i} = 1, \ldots, 5$), $\mathbf{J_j}(e,\dot{e})$, ($\mathbf{j} = 1, \ldots, 4$) and $\mathbf{H_j}(e,\dot{e})$, ($\mathbf{j} = 1, \ldots, 4$):

(6) $\mathbf{A_i}(e,\dot{e}) \equiv \int_0^{2\pi} [1 + (e - \dot{e})cos\varphi]^{-\mathrm{i}}\, d\varphi$; ($\mathbf{i} = 1, \ldots, 5$),

(7) $\mathbf{J_j}(e,\dot{e}) \equiv \int_0^{2\pi} (1 + ecos\varphi)^{-1}[1 + (e - \dot{e})cos\varphi]^{-\mathrm{j}}\, d\varphi$; ($\mathbf{j} = 1, \ldots, 4$),

(8) $\mathbf{H_j}(e,\dot{e}) \equiv \int_0^{2\pi} (1 + ecos\varphi)^{-\mathrm{j}}[1 + (e - \dot{e})cos\varphi]^{-1}\, d\varphi$; ($\mathbf{j} = 1, \ldots, 4$).

The analytical evaluations of the later three integrals (6) – (8) were derived and discussed in paper [3]. It is worth to note, that the integration over the azimuthal angle φ in the above written integrals (1) – (8) is a consequence of the angle averaging over the each particle orbit in the interval $\varphi \in [0, 2\pi]$. We have also to stress, that the applied in the next chapters approaches for analytical evaluations of the integrals $\mathbf{L_i}(e,\dot{e})$ and $\mathbf{K_j}(e,\dot{e})$ are useful for higher integer values of the powers of the denominator $[1 + (e - \dot{e})cos\varphi]$. That is to say, for $\mathbf{i} > 3$, or $\mathbf{j} > 4$. But we shall restrict us only to those generality levels, which are enough to solve the considered by us particular problem of analyzing the dynamical equation of the *stationary* elliptical accretion discs. We do not set ourselves as an object to solve the complete mathematical task for all **i** and **j**.

2. Analytical computation of the integral $\mathbf{K_2}(e,\dot{e}) \equiv \int_0^{2\pi}[ln(1 + ecos\varphi)][1 + (e - \dot{e})cos\varphi]^{-2}\, d\varphi$

It is appropriate to begin the evaluation of the integrals $\mathbf{K_j}(e,\dot{e})$, ($\mathbf{j}$ = = 2, …, 5) (using correlation relations), from the integral $\mathbf{K_2}(e,\dot{e})$, not from the integral $\mathbf{K_5}(e,\dot{e})$. Hence, according to the definitions (2):

(9)
$$\mathbf{K_2}(e,\dot{e}) \equiv \int_0^{2\pi}[ln(1 + ecos\varphi)][1 + (e - \dot{e})cos\varphi]^{-2}\, d\varphi =$$
$$= -(e - \dot{e})^{-2}\int_0^{2\pi}[1 - (e - \dot{e})^2cos^2\varphi][ln(1 + ecos\varphi)][1 + (e - \dot{e})cos\varphi]^{-2}\, d\varphi +$$
$$+ (e - \dot{e})^{-2}\int_0^{2\pi}[ln(1 + ecos\varphi)][1 + (e - \dot{e})cos\varphi]^{-2}\, d\varphi -$$
$$- (e - \dot{e})^{-1}\int_0^{2\pi}(sin\varphi)[ln(1 + ecos\varphi)][1 + (e - \dot{e})cos\varphi]^{-2}\, d[1 + (e - \dot{e})cos\varphi] =$$
$$= -(e - \dot{e})^{-2}\mathbf{K_1}(e,\dot{e}) + (e - \dot{e})^{-2}\int_0^{2\pi}ln(1 + ecos\varphi)\, d\varphi - (e - \dot{e})^{-2}\mathbf{K_1}(e,\dot{e}) + (e - \dot{e})^{-2}\mathbf{K_2}(e,\dot{e}) -$$
$$- (e - \dot{e})^{-2}\int_0^{2\pi}\{[1 + (e - \dot{e})cos\varphi] - 1\}[1 + (e - \dot{e})cos\varphi]^{-1}[ln(1 + ecos\varphi)]\, d\varphi +$$
$$+ [e/(e - \dot{e})]\int_0^{2\pi}(1 - cos^2\varphi)(1 + ecos\varphi)^{-1}[1 + (e - \dot{e})cos\varphi]^{-1}\, d\varphi =$$
$$= (e - \dot{e})^{-2}[\mathbf{K_2}(e,\dot{e}) - \mathbf{K_1}(e,\dot{e})] + \{(e^2 - 1)/[e(e - \dot{e})]\}\mathbf{J_1}(e,\dot{e}) + [e/(e - \dot{e})]\mathbf{A_1}(e,\dot{e}) -$$
$$- (e - \dot{e})^{-2}\int_0^{2\pi}\{[1 + (e - \dot{e})cos\varphi] - 1\}[1 + (e - \dot{e})cos\varphi]^{-1}\, d\varphi = (e - \dot{e})^{-2}[\mathbf{K_2}(e,\dot{e}) - \mathbf{K_1}(e,\dot{e})] +$$
$$+ \{(e^2 - 1)/[e(e - \dot{e})]\}\mathbf{J_1}(e,\dot{e}) + [e(e - \dot{e})]^{-1}\mathbf{A_1}(e,\dot{e}) - 2\pi(e - \dot{e})^{-2} + (e - \dot{e})^{-2}\mathbf{A_1}(e,\dot{e}).$$

This relation gives the final expression for the integral $\mathbf{K_2}(e,\dot{e})$.Using the already computed expressions for $\mathbf{K_1}(e,\dot{e})$ (see equality (5)), $\mathbf{J_1}(e,\dot{e})$ (see equality (27) from paper [3]) and $\mathbf{A_1}(e,\dot{e})$ (see equality (7) from paper [3]), we are in a position to write the explicit **analytical** solution for $\mathbf{K_2}(e,\dot{e})$. Finally, we obtain that:

(10)
$$\mathbf{K_2}(e,\dot{e}) = 2\pi[1 - (e - \dot{e})^2]^{-3/2}ln\mathbf{\{}\{2 - 3e^2 + e^4 + 3e\dot{e} - 2e^3\dot{e} - \dot{e}^2 + e^2\dot{e}^2 +$$
$$+ (-2 + 2e^2 - 3e\dot{e} + \dot{e}^2)(1 - e^2)^{1/2} + (-2 + 2e^2 - e\dot{e})[1 - (e - \dot{e})^2]^{1/2} +$$
$$+ (2 - e^2 + e\dot{e})(1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2}\}(e - \dot{e})^{-2}[1 - (1 - e^2)^{1/2}]^{-1}\mathbf{\}} - 2\pi e\dot{e}^{-1}[1 - (e - \dot{e})^2]^{-1/2} +$$
$$+ 2\pi(e - e^3 - \dot{e} + e^2\dot{e})\dot{e}^{-1}(1 - e^2)^{-1/2}[1 - (e - \dot{e})^2]^{-1} + 2\pi[1 - (e - \dot{e})^2]^{-1}.$$

We shall not perform here the tedious algebraic computations, proving in a rigorous mathematical manner, that the written above analytical solution (10) for the integral $\mathbf{K_2}(e,\dot{e})$ remains valid even in the cases, when

$e(u)$, and/or $e(u) - \dot{e}(u)$ are equal to zero. These particular values are preliminary excluded in the derivation of the equality (10). Because they enter as factors into the denominators of some of the intermediate terms. Correspondingly, this situation leads to the necessity to examine the indicated cases in a separate way. The resolving of the designated problem may follow the analogous procedure, which is called to overcome such difficulties, appearing under the analytical solving of the integrals $\mathbf{K_1}(e,\dot{e})$, $\mathbf{L_i}(e,\dot{e})$, ($\mathbf{i}$ = 0, …, 3), $\mathbf{A_i}(e,\dot{e})$, ($\mathbf{i}$ = 1,…, 5), $\mathbf{J_j}(e,\dot{e})$, ($\mathbf{j}$ =1, …, 4), $\mathbf{H_j}(e,\dot{e})$, ($\mathbf{j}$ = = 1, …, 4) (see papers [3] and [4]). Especially, the used approach is based on the application of the L'Hospital's rule for resolving of indeterminacies of the type 0/0. We shall return later in this paper to the arising problem.

3. Analytical computation of the integral $\mathbf{K_3}(e,\dot{e}) \equiv \int_0^{2\pi}[\ln(1 + e\cos\varphi)][1 + (e - \dot{e})\cos\varphi]^{-3}\, d\varphi$

Let us compute the integral $\mathbf{K_3}(e,\dot{e})$. According to the definition (2):

(11)
$$\mathbf{K_3}(e,\dot{e}) = -(e-\dot{e})^{-2}\int_0^{2\pi}[1-(e-\dot{e})^2\cos^2\varphi][\ln(1+e\cos\varphi)][1+(e-\dot{e})\cos\varphi]^{-3}\,d\varphi +$$
$$+(e-\dot{e})^{-2}\int_0^{2\pi}[\ln(1+e\cos\varphi)][1+(e-\dot{e})\cos\varphi]^{-3}\,d\varphi -$$
$$-(e-\dot{e})^{-1}\int_0^{2\pi}(\sin\varphi)[\ln(1+e\cos\varphi)][1+(e-\dot{e})\cos\varphi]^{-3}\,d[1+(e-\dot{e})\cos\varphi] =$$
$$= -(e-\dot{e})^{-2}\mathbf{K_2}(e,\dot{e}) + (e-\dot{e})^{-2}\int_0^{2\pi}\{[1+(e-\dot{e})\cos\varphi]-1\}[\ln(1+e\cos\varphi)][1+(e-\dot{e})\cos\varphi]^{-2}\,d\varphi +$$
$$+(e-\dot{e})^{-2}\mathbf{K_3}(e,\dot{e}) + [2(e-\dot{e})]^{-1}\{(\sin\varphi)[\ln(1+e\cos\varphi)][1+(e-\dot{e})\cos\varphi]^{-2}\Big|_0^{2\pi} -$$
$$-\int_0^{2\pi}(\cos\varphi)[\ln(1+e\cos\varphi)][1+(e-\dot{e})\cos\varphi]^{-2}\,d\varphi + e\int_0^{2\pi}(\sin^2\varphi)(1+e\cos\varphi)^{-1}[1+(e-\dot{e})\cos\varphi]^{-2}\,d\varphi\} =$$
$$= -(e-\dot{e})^{-2}\mathbf{K_2}(e,\dot{e}) + (e-\dot{e})^{-2}\mathbf{K_1}(e,\dot{e}) - (e-\dot{e})^{-2}\mathbf{K_2}(e,\dot{e}) + (e-\dot{e})^{-2}\mathbf{K_3}(e,\dot{e}) - [2(e-\dot{e})^2]^{-1}\mathbf{K_1}(e,\dot{e}) +$$
$$+[2(e-\dot{e})^2]^{-1}\mathbf{K_2}(e,\dot{e}) + e[2(e-\dot{e})]^{-1}\int_0^{2\pi}(1+e\cos\varphi)^{-1}[1+(e-\dot{e})\cos\varphi]^{-2}\,d\varphi +$$
$$+[2e(e-\dot{e})]^{-1}\int_0^{2\pi}[(1-e^2\cos^2\varphi)-1](1+e\cos\varphi)^{-1}[1+(e-\dot{e})\cos\varphi]^{-2}\,d\varphi =$$
$$= (e-\dot{e})^{-2}[\mathbf{K_3}(e,\dot{e}) - (3/2)\mathbf{K_2}(e,\dot{e}) + (1/2)\mathbf{K_1}(e,\dot{e})] - (1-e^2)[2e(e-\dot{e})]^{-1}\{\int_0^{2\pi}[1+(e-\dot{e})\cos\varphi]^{-2}\,d\varphi -$$
$$-[e/(e-\dot{e})]\int_0^{2\pi}\{[1+(e-\dot{e})\cos\varphi]-1\}(1+e\cos\varphi)^{-1}[1+(e-\dot{e})\cos\varphi]^{-2}\,d\varphi\} +$$

$$+ [2e(e - \dot{e})]^{-1}\int_0^{2\pi}[1 + (e - \dot{e})cos\varphi]^{-2}\, d\varphi - [2(e - \dot{e})^2]^{-1}\int_0^{2\pi}\{[1 + (e - \dot{e})cos\varphi] - 1\}[1 + (e - \dot{e})cos\varphi]^{-2}\, d\varphi =$$

$$= (e - \dot{e})^{-2}\{\mathbf{K_3}(e,\dot{e}) - (3/2)\mathbf{K_2}(e,\dot{e}) + (1/2)\mathbf{K_1}(e,\dot{e}) + [(1 + e^2 - e\dot{e})/2]\mathbf{A_2}(e,\dot{e}) +$$
$$+ [(1 - e^2)/2][\mathbf{J_1}(e,\dot{e}) - \mathbf{J_2}(e,\dot{e})] - (1/2)\mathbf{A_1}(e,\dot{e})\}.$$

Transferring the *unknown* function $\mathbf{K_3}(e,\dot{e})$ from the right-hand-side to the left one into the above relation, we can write $\mathbf{K_3}(e,\dot{e})$ through the *already known* functions:

(12) $$\mathbf{K_3}(e,\dot{e}) = \{2[1 - (e - \dot{e})^2]\}^{-1}\{3\mathbf{K_2}(e,\dot{e}) - \mathbf{K_1}(e,\dot{e}) - (1 + e^2 - e\dot{e})\mathbf{A_2}(e,\dot{e}) +$$
$$+ \mathbf{A_1}(e,\dot{e}) + (1 - e^2)[\mathbf{J_2}(e,\dot{e}) - \mathbf{J_1}(e,\dot{e})]\}.$$

In the above formula, functions $\mathbf{K_1}(e,\dot{e})$ and $\mathbf{K_2}(e,\dot{e})$ are given by the analytical solutions (5) and (10), respectively. The other integrals in the right-hand-side of (12) are evaluated in the earlier paper [3], as follows: $\mathbf{A_1}(e,\dot{e})$ – formula (7), $\mathbf{A_2}(e,\dot{e})$ – formula (8), $\mathbf{J_1}(e,\dot{e})$ – formula (27) and $\mathbf{J_2}(e,\dot{e})$ – formula (34). The numerations of the later formulas correspond to [3]. The substitution of these expressions into (12) gives the explicit form of the solution (12) as a function of the eccentricity $e(u)$ and its derivative $\dot{e}(u)$. After some simple algebraic transformations, we are in a position to write the final analytical solution for the integral $\mathbf{K_3}(e,\dot{e})$:

(13) $$\mathbf{K_3}(e,\dot{e}) = \pi[1 - (e - \dot{e})^2]\}^{-5/2}(2 + e^2 - 2e\dot{e} + \dot{e}^2)ln\mathbf{\{}\{2 - 3e^2 + e^4 + 3e\dot{e} - 2e^3\dot{e} - \dot{e}^2 + e^2\dot{e}^2 +$$
$$+ (-2 + 2e^2 - 3e\dot{e} + \dot{e}^2)(1 - e^2)^{1/2} + (-2 + 2e^2 - e\dot{e})[1 - (e - \dot{e})^2]^{1/2} +$$
$$+ (2 - e^2 + e\dot{e})(1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2}\}(e - \dot{e})^{-2}[1 - (1 - e^2)^{1/2}]^{-1}\mathbf{\}} +$$
$$+ \pi(-e^2 + 2e^4 - e^6 - 2e\dot{e} - 2e^3\dot{e} + 4e^5\dot{e} - 2e^2\dot{e}^2 - 6e^4\dot{e}^2 + 2e\dot{e}^3 + 4e^3\dot{e}^3 - e^2\dot{e}^4)\dot{e}^{-2}[1 - (e - \dot{e})^2]^{-5/2} +$$
$$+ \pi(e^2 - 2e^4 + e^6 + 2e\dot{e} + e^3\dot{e} - 3e^5\dot{e} - 3\dot{e}^2 + 3e^4\dot{e}^2 + e\dot{e}^3 - e^3\dot{e}^3)\dot{e}^{-2}(1 - e^2)^{-1/2}[1 - (e - \dot{e})^2]^{-2} +$$
$$+ 3\pi[1 - (e - \dot{e})^2]^{-2}.$$

4. Analytical expressions for the integrals $\mathbf{K_4}(e,\dot{e}) \equiv \int_0^{2\pi}[ln(1 + ecos\varphi)][1 + (e - \dot{e})cos\varphi]^{-4}\, d\varphi$ and $\mathbf{K_5}(e,\dot{e}) \equiv \int_0^{2\pi}[ln(1 + ecos\varphi)][1 + (e - \dot{e})cos\varphi]^{-5}\, d\varphi$

The computational procedure of the integrals $\mathbf{K_4}(e,\dot{e})$ and $\mathbf{K_5}(e,\dot{e})$ exactly resembles to that, which we described earlier in details, when we solved the integrals $\mathbf{K_2}(e,\dot{e})$ and $\mathbf{K_3}(e,\dot{e})$. As it can be established from these calculations, the applied approach is to develop the integrands from the corresponding definitions (2) in such a way that into the right-hand-side to appear the same integral, multiplied by a factor different from unity. The later condition is crucial for the method of computation to work, because the integral, for which we are seeking, may be transferred into the left-hand-side of the equality. The result will be that in the left we shall have only the unknown integral, multiplied by a factor different from zero. Into the right-hand-side remain integrals of the same type (2), but with index **j** less then that of the integral under resolving. The formers are already solved. This is essentially a recurrent procedure. Of course, into the right-hand-side also present integrals of the types $\mathbf{A_i}(e,\dot{e})$ (6) and $\mathbf{J_j}(e,\dot{e})$ (7), but their analytical expressions are successfully computed in an earlier paper [3]. Therefore, it is only a matter of tedious algebra to resolve analytically the integrals $\mathbf{K_4}(e,\dot{e})$ and $\mathbf{K_5}(e,\dot{e})$, starting directly from their definitions (2). For such reasons, we were motivated to skip here the detailed writing (as we have already done for $\mathbf{K_2}(e,\dot{e})$ and $\mathbf{K_3}(e,\dot{e})$) of the intermediate steps, leading to the solutions of $\mathbf{K_4}(e,\dot{e})$ and $\mathbf{K_5}(e,\dot{e})$. We shall give only their expressions through the integrals $\mathbf{K_j}(e,\dot{e})$, ($\mathbf{j}$ = 1, 2, 3), $\mathbf{A_i}(e,\dot{e})$, ($\mathbf{i}$ = 2, 3, 4) and $\mathbf{J_j}(e,\dot{e})$, ($\mathbf{j}$ = 2, 3, 4), and the corresponding final analytical formulas for them. We underline that the analytical solution for $\mathbf{K_j}(e,\dot{e})$, ($\mathbf{j}$ = 1, …, 5), $\mathbf{L_i}(e,\dot{e})$, ($\mathbf{i}$ = =1, 2, 3), $\mathbf{A_i}(e,\dot{e})$, ($\mathbf{i}$ = 1, …, 5), $\mathbf{J_j}(e,\dot{e})$, ($\mathbf{j}$ = 1, …, 4) and $\mathbf{H_j}(e,\dot{e})$, ($\mathbf{j}$ = 1, …, 4) are also tested, by means of numerical methods, for a dense enough two-dimensional lattice with respect to (e, $\dot{e}$). In conclusion, we write down the following results:

(14) $$\mathbf{K_4}(e,\dot{e}) = \{3[1-(e-\dot{e})^2]\}^{-1}\{5\mathbf{K_3}(e,\dot{e}) - 2\mathbf{K_2}(e,\dot{e}) - (1+e^2-e\dot{e})\mathbf{A_3}(e,\dot{e}) + \mathbf{A_2}(e,\dot{e}) + (1-e^2)[\mathbf{J_3}(e,\dot{e}) - \mathbf{J_2}(e,\dot{e})]\}.$$

It remains to substitute the corresponding analytical expressions for $\mathbf{K_3}(e,\dot{e})$ (formula (13)), $\mathbf{K_2}(e,\dot{e})$ (formula (10)), $\mathbf{A_2}(e,\dot{e})$ (formula (8) from [3]), $\mathbf{A_3}(e,\dot{e})$ (formula (16) from [3]), $\mathbf{J_2}(e,\dot{e})$ (formula (34) from [3]) and $\mathbf{J_3}(e,\dot{e})$ (formula (42) from [3]). The conclusive result from such complicated evaluation can be written as follows:

(15) $$\begin{aligned}\mathbf{K_4}(e,\dot{e}) &= \pi(2+3e^2-6e\dot{e}+3\dot{e}^2)[1-(e-\dot{e})^2]^{-7/2}ln\boldsymbol{\{}\{2-3e^2+e^4+3e\dot{e}-2e^3\dot{e}-\dot{e}^2+e^2\dot{e}^2+\\ &+(-2+2e^2-3e\dot{e}+\dot{e}^2)(1-e^2)^{1/2}+(-2+2e^2-e\dot{e})[1-(e-\dot{e})^2]^{1/2}+\\ &+(2-e^2+e\dot{e})(1-e^2)^{1/2}[1-(e-\dot{e})^2]^{1/2}\}(e-\dot{e})^{-2}[1-(1-e^2)^{1/2}]^{-1}\boldsymbol{\}}+\\ &+(\pi/3)(-2e^3+4e^5-2e^7-3e^2\dot{e}-5e^4\dot{e}+8e^6\dot{e}-6e\dot{e}^2-5e^3\dot{e}^2-12e^5\dot{e}^2-2\dot{e}^3+\\ &+9e^2\dot{e}^3+8e^4\dot{e}^3-3e\dot{e}^4-2e^3\dot{e}^4)\dot{e}^{-3}[1-(e-\dot{e})^2]^{-5/2}+(\pi/3)(2e^3-6e^5+6e^7-2e^9+3e^2\dot{e}+4e^4\dot{e}-\\ &-17e^6\dot{e}+10e^8\dot{e}+6e\dot{e}^2+e^3\dot{e}^2+13e^5\dot{e}^2-20e^7\dot{e}^2-11\dot{e}^3-12e^2\dot{e}^3+3e^4\dot{e}^3+20e^6\dot{e}^3+17e\dot{e}^4-\\ &-7e^3\dot{e}^4-10e^5\dot{e}^4-4\dot{e}^5+2e^2\dot{e}^5+2e^4\dot{e}^5)\dot{e}^{-3}(1-e^2)^{-1/2}[1-(e-\dot{e})^2]^{-3}+(2\pi/3)[1-(e-\dot{e})^2]^{-5/2}+\\ &+(\pi/3)(11+4e^2-8e\dot{e}+4\dot{e}^2)[1-(e-\dot{e})^2]^{-3}.\end{aligned}$$

The explicit form of the integral $\mathbf{K_5}(e,\dot{e})$, as a function of the eccentricity $e(u)$ and its derivative $\dot{e}(u)$, may be written in a similar way. At first, the direct processing of the definition (2) for the integral $\mathbf{K_5}(e,\dot{e})$ leads to the intermediate evaluation for $\mathbf{K_5}(e,\dot{e})$, analogous to the relation (14) for $\mathbf{K_4}(e,\dot{e})$:

(16) $$\mathbf{K_5}(e,\dot{e}) = \{4[1-(e-\dot{e})^2]\}^{-1}[7\mathbf{K_4}(e,\dot{e}) - 3\mathbf{K_3}(e,\dot{e}) + (e-\dot{e})(1-e^2)e^{-1}\mathbf{J_4}(e,\dot{e}) + \mathbf{A_3}(e,\dot{e}) - (2e-\dot{e})e^{-1}\mathbf{A_4}(e,\dot{e})].$$

Like to the previous case above, the substitution into the relation (16) of the analytical solutions for $\mathbf{K_4}(e,\dot{e})$ (formula (15)), $\mathbf{K_3}(e,\dot{e})$ (formula (13)), $\mathbf{A_3}(e,\dot{e})$ (formula (16) from [3]), $\mathbf{A_4}(e,\dot{e})$ (formula (9) from [3]) and

$\mathbf{J_4}(e,\dot{e})$ (formula (47) from [3]), gives, after some tedious algebra, the final explicit analytical evaluation:

(17)
$$\mathbf{K_5}(e,\dot{e}) = (\pi/4)(8 + 24e^2 + 3e^4 - 48e\dot{e} - 12e^3\dot{e} + 24\dot{e}^2 + 18e^2\dot{e}^2 - 12e\dot{e}^3 + 3\dot{e}^4)[1 - (e - \dot{e})^2]^{-9/2}\times$$
$$\times ln\{\{2 - 3e^2 + e^4 + 3e\dot{e} - 2e^3\dot{e} - \dot{e}^2 + e^2\dot{e}^2 + (-2 + 2e^2 - 3e\dot{e} + \dot{e}^2)(1 - e^2)^{1/2} +$$
$$+ (-2 + 2e^2 - e\dot{e})[1 - (e - \dot{e})^2]^{1/2} + (2 - e^2 + e\dot{e})(1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2}\}(e - \dot{e})^{-2}\times$$
$$\times[1 - (1 - e^2)^{1/2}]^{-1}\} + (\pi/6)(-3e^4 + 9e^6 - 9e^8 + 3e^{10} - 4e^3\dot{e} - 10e^5\dot{e} + 32e^7\dot{e} - 18e^9\dot{e} - 6e^2\dot{e}^2 - 4e^4\dot{e}^2 -$$
$$- 35e^6\dot{e}^2 + 45e^8\dot{e}^2 - 12e\dot{e}^3 - 16e^3\dot{e}^3 - 60e^7\dot{e}^3 + 39e^2\dot{e}^4 + 25e^4\dot{e}^4 + 45e^6\dot{e}^4 - 18e\dot{e}^5 - 16e^3\dot{e}^5 - 18e^5\dot{e}^5 +$$
$$+ 3e^2\dot{e}^6 + 3e^4\dot{e}^6)\dot{e}^{-4}[1 - (e - \dot{e})^2]^{-7/2} + (\pi/12)(6e^4 - 24e^6 + 36e^8 - 24e^{10} + 6e^{12} + 8e^3\dot{e} + 18e^5\dot{e} -$$
$$- 102e^7\dot{e} + 118e^9\dot{e} - 42e^{11}\dot{e} + 12e^2\dot{e}^2 + e^4\dot{e}^2 + 88e^6\dot{e}^2 - 227e^8\dot{e}^2 + 126e^{10}\dot{e}^2 + 24e\dot{e}^3 + 16e^3\dot{e}^3 -$$
$$- 35e^5\dot{e}^3 + 205e^7\dot{e}^3 - 210e^9\dot{e}^3 - 50\dot{e}^4 - 138e^2\dot{e}^4 + 48e^4\dot{e}^4 - 70e^6\dot{e}^4 + 210e^8\dot{e}^4 + 182e\dot{e}^5 - 40e^3\dot{e}^5 -$$
$$- 16e^5\dot{e}^5 - 126e^7\dot{e}^5 - 55\dot{e}^6 - 4e^2\dot{e}^6 + 17e^4\dot{e}^6 + 42e^6\dot{e}^6 + 9e\dot{e}^7 - 3e^3\dot{e}^7 - 6e^5\dot{e}^7)\dot{e}^{-4}\times$$
$$\times(1 - e^2)^{-1/2}[1 - (e - \dot{e})^2]^{-4} + (5\pi/12)(10 + 11e^2 - 22e\dot{e} + 11\dot{e}^2)[1 - (e - \dot{e})^2]^{-4}.$$

5. Expressions for the integrals $\mathbf{K_i}(e,\dot{e}) \equiv \int_0^{2\pi}[ln(1 + e\cos\varphi)][1 + (e - \dot{e})\cos\varphi]^{-\mathbf{i}}\, d\varphi$, (i = 1, …, 5) for some specific values of their arguments $e(u)$ and $\dot{e}(u)$

Let us, at first, introduce a useful notation, with a view to shorten (*in some cases*) the writing of the analytical formulas. More specially, we intent to denote with the function $\mathbf{Z}(e,\dot{e})$ the argument of the logarithmic function, which enters both into the intermediate calculations and the final solutions for the integrals $\mathbf{K_i}(e,\dot{e})$, (**i** = 1, …, 5). This argument *is the same for all* **i** = = 1, …, 5, which makes it reasonable to introduce into use one more notation into the system of notations, used in the present paper. We stress, however, that we shall skip such a shortening of the notations (as we already have done until now), if we want to write, as possible, but more tedious, in the "*most explicit*" form the dependence of the expressions on $e(u)$ and $\dot{e}(u)$.

Therefore, we define that:

(18)
$$\mathbf{Z}(e,\dot{e}) \equiv \{2 - 3e^2 + e^4 + 3e\dot{e} - 2e^3\dot{e} - \dot{e}^2 + e^2\dot{e}^2 + (-2 + 2e^2 - 3e\dot{e} + \dot{e}^2)(1 - e^2)^{1/2} +$$
$$+ (-2 + 2e^2 - e\dot{e})[1 - (e - \dot{e})^2]^{1/2} +$$
$$+ (2 - e^2 + e\dot{e})(1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2}\}(e - \dot{e})^{-2}[1 - (1 - e^2)^{1/2}]^{-1}.$$

We shall write the above function for two different pairs of its arguments, namely: $\{e,\ \dot{e} = 0\}$ and $\{e - \dot{e},\ -\dot{e}\}$. These combinations will arise during the further use of the notation formula (18):

(19)
$$\mathbf{Z}(e,\dot{e} = 0) = 2[2 - 3e^2 + e^4 - 2(1 - e^2)^{3/2}]e^{-2}[1 - (1 - e^2)^{1/2}]^{-1},$$

(20)
$$\mathbf{Z}(e - \dot{e}, -\dot{e}) = \{2 - 3e^2 + e^4 + 3e\dot{e} - 2e^3\dot{e} - \dot{e}^2 + e^2\dot{e}^2 + (-2 + 2e^2 - 3e\dot{e} + \dot{e}^2)(1 - e^2)^{1/2} +$$
$$+ (-2 + 2e^2 - e\dot{e})[1 - (e - \dot{e})^2]^{1/2} + (2 - e^2 + e\dot{e})(1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2}\}e^{-2}\{1 - [1 - (e -$$
$$- \dot{e})^2]^{1/2}\}^{-1}.$$

Until now, we have computed analytically $\mathbf{K_i}(e,\dot{e})$, (**i** = 1, …, 5) under the condition $\dot{e}(u) \neq 0$. For **i** = 2, 3, 4 and 5 the derivative $\dot{e}(u)$ of the eccentricity $e(u)$ presents as a factor in the denominators of some of the summands in these expressions. Therefore, it is not reasonable to calculate

$\mathbf{K_i}(e,0)$, ($\mathbf{i}$ = 2, …, 5) simply setting $\dot{e}(u) = 0$ into the already calculated results for this functions, valid for $\dot{e}(u) \neq 0$. Of course, we may attempt to use the L'Hospital's rule for evaluating of indeterminacies of the type 0/0, but this approach (if it works) will probably be too tedious for our exposition. We shall use the same computational scheme for evaluating of the integrals $\mathbf{K_i}(e,0)$, ($\mathbf{i}$ = 2, …, 5), as in the previous cases, when $\dot{e}(u) \neq 0$. But the recurrence formulas will now be established for the specific case $\dot{e}(u) = 0$. Before to begin with this procedure, we note that the just discussed problem, concerning the nullification of $\dot{e}(u)$, does not play a role in deriving of $\mathbf{K_1}(e,0)$ from the expression (5) for $\mathbf{K_1}(e,\dot{e})$. We are able, without any contradictions, to set $\dot{e}(u) = 0$, to obtain that:

(21) $$\mathbf{K_1}(e,0) = 2\pi(1-e^2)^{-1/2} ln\{[2-3e^2+e^4-2(1-e^2)^{3/2}-2(1-e^2)^{3/2} + (2-e^2)(1-e^2)]e^{-2}[1-(1-e^2)^{1/2}]^{-1}\} = 2\pi(1-e^2)^{-1/2} ln\{2[2-3e^2+e^4-2(1-e^2)^{3/2}]\times e^{-2}[1-(1-e^2)^{1/2}]^{-1}\} = 2\pi(1-e^2)^{-1/2} ln\mathbf{Z}(e,0).$$

It is easily seen that $lim\mathbf{K_1}(e,0) \to 0$, when $e(u)$ approaches zero.

Now, we start to the *direct* evaluation of the integral $\mathbf{K_2}(e,0)$:

(22) $$\mathbf{K_2}(e,0) = \int_0^{2\pi}(cos^2\varphi + sin^2\varphi)[ln(1+ecos\varphi)](1+ecos\varphi)^{-2}\,d\varphi =$$
$$= -e^{-2}\int_0^{2\pi}(1-e^2cos^2\varphi)[ln(1+ecos\varphi)](1+ecos\varphi)^{-2}\,d\varphi + e^{-2}\int_0^{2\pi}[ln(1+ecos\varphi)](1+ecos\varphi)^{-2}d\varphi -$$
$$- e^{-1}\int_0^{2\pi}(sin\varphi)[ln(1+ecos\varphi)](1+ecos\varphi)^{-2}\,d(1+ecos\varphi) =$$
$$= -e^{-2}\mathbf{K_1}(e,0) + e^{-2}\mathbf{K_0}(e) - e^{-2}\mathbf{K_1}(e,0) + e^{-2}\mathbf{K_2}(e,0) -$$
$$- e^{-2}\int_0^{2\pi}[(1+ecos\varphi)-1][ln(1+ecos\varphi)](1+ecos\varphi)^{-1}\,d\varphi + \int_0^{2\pi}(1-cos^2\varphi)(1+ecos\varphi)^{-2}\,d\varphi =$$
$$= e^{-2}\mathbf{K_2}(e,0) - 2e^{-2}\mathbf{K_1}(e,0) + e^{-2}\mathbf{K_0}(e) - e^{-2}\mathbf{K_0}(e) + e^{-2}\mathbf{K_1}(e,0) + \mathbf{J_1}(e,0) +$$
$$+ e^{-2}\int_0^{2\pi}[(1-e^2cos^2\varphi)-1](1+ecos\varphi)^{-2}\,d\varphi =$$
$$= -e^{-2}\mathbf{J_1}(e,0) + e^{-2}\mathbf{K_2}(e,0) - e^{-2}\mathbf{K_1}(e,0) + \mathbf{J_1}(e,0) + e^{-2}\mathbf{A_1}(e,0) - 2\pi/e^2 + e^{-2}\mathbf{A_1}(e,0) =$$
$$= e^{-2}\mathbf{K_2}(e,0) - e^{-2}\mathbf{K_1}(e,0) + 2e^{-2}\mathbf{A_1}(e,0) + (e^2-1)e^{-2}\mathbf{J_1}(e,0) - 2\pi/e^2.$$

This relation gives a possibility to find an analytical solution for the unknown function $\mathbf{K_2}(e,0)$, because the other functions of the eccentricity $e(u)$ are already known: $\mathbf{K_1}(e,0)$ from formula (21), $\mathbf{A_1}(e,0)$ (formula (20) from paper [3]), and $\mathbf{J_1}(e,0)$ (formula (35) from paper [3]). Taking into account these relations, we obtain:

(23) $$\mathbf{K_2}(e,0) = (1-e^2)^{-1}[\mathbf{K_1}(e,0) - 2\pi(1-e^2)^{-1/2} + 2\pi] =$$
$$= (1-e^2)^{-1}[2\pi(1-e^2)^{-1/2} ln\mathbf{Z}(e,0) - 2\pi(1-e^2)^{-1/2} + 2\pi] =$$
$$= 2\pi(1-e^2)^{-3/2}\{ln\{2[2-3e^2+e^4-2(1-e^2)^{3/2}]e^{-2}[1-(1-e^2)^{1/2}]^{-1}\} - 1 + (1-e^2)^{1/2}\}.$$

We underline that this result does not require the condition $\dot{e}(u) \neq 0$. Just the opposite is true! At the very beginning of the computations, we set $\dot{e}(u) = 0$. We shall not write here the proof that the relation (10) for $\mathbf{K_2}(e,\dot{e})$ (derived under the suppositions $e(u) \neq 0$, $\dot{e}(u) \neq 0$ and $e(u) - \dot{e}(u) \neq 0$) in the limit $\dot{e}(u) \to 0$ coincides with the above relation (23) for $\mathbf{K_2}(e,0)$. This statement can easily be checked by the means of the L'Hospital's rule. Finally, we note that the transition $e(u) \to 0$ in the formula (23) leads to a vanishing result: $\mathbf{K_2}(0,0) = 0$, which corresponds to the expected value from the definition (9). Despite of the our skipping of the detailed considerations of the behavior of the established relations, under the transitions $e(u) \to 0$, $\dot{e}(u) \to 0$ and $[e(u) - \dot{e}(u)] \to 0$, we nevertheless stress that such considerations are important. They give a certainty that the transitions through these singular points are continuous. Such a detailed treatment was done for the integrals $\mathbf{A_i}(e,\dot{e})$, ($\mathbf{i} = 1, \ldots, 5$), $\mathbf{J_j}(e,\dot{e})$, ($\mathbf{j} = 1, \ldots, 4$) and $\mathbf{H_j}(e,\dot{e})$, ($\mathbf{j} = 1, \ldots, 4$) in paper [3]. But with a view to give a shorter description of the procedures of solving of the integrals $\mathbf{K_i}(e,\dot{e})$, ($\mathbf{i} = 1, \ldots, 5$), we do not write out such tedious computations, introducing the use of the L'Hospital's rule for resolving of indeterminacies of the type 0/0. Before to proceed further, we emphasize that the above-mentioned remarks, concerning the solution of the integral $\mathbf{K_2}(e,\dot{e})$, are also remaining valid for the solutions of the "higher order" integrals $\mathbf{K_i}(e,\dot{e})$, ($\mathbf{i} = 3, 4, 5$). We now begin with the description of their computation, and *we shall not return to the discussion of such similar matter later*.

(24)
$$\mathbf{K_3}(e,0) = \int_0^{2\pi}(cos^2\varphi + sin^2\varphi)[ln(1 + ecos\varphi)](1 + ecos\varphi)^{-3}\, d\varphi =$$

$$= -e^2\int_0^{2\pi}(1 - e^2cos^2\varphi)[ln(1 + ecos\varphi)](1 + ecos\varphi)^{-3}\, d\varphi + e^{-2}\int_0^{2\pi}[ln(1 + ecos\varphi)](1 + ecos\varphi)^{-3}\, d\varphi -$$

$$- e^{-1}\int_0^{2\pi}(sin\varphi)[ln(1 + ecos\varphi)](1 + ecos\varphi)^{-3}\, d(1 + ecos\varphi) = \ldots =$$

$$= [2(1 - e^2)]^{-1}\{3\mathbf{K_2}(e,0) - \mathbf{K_1}(e,0) - (1 + e^2)\mathbf{A_2}(e,0) + \mathbf{A_1}(e,0) + (1 - e^2)[\mathbf{J_2}(e,0) - \mathbf{J_1}(e,0)]\} =$$
$$= [2(1 - e^2)]^{-1}\{6\pi(1 - e^2)^{-3/2}[ln\mathbf{Z}(e,0) - 1 + (1 - e^2)^{1/2}] - 2\pi(1 - e^2)(1 - e^2)^{-3/2}ln\mathbf{Z}(e,0) -$$
$$- 2\pi(1 + e^2)(1 - e^2)^{-3/2} + 2\pi(1 - e^2)(1 - e^2)^{-3/2} + (1 - e^2)[\pi(2 + e^2)(1 - e^2)^{-5/2} -$$
$$- 2\pi(1 - e^2)(1 - e^2)^{-5/2}]\}.$$

In the above equality the symbol "… ="denotes some of the skipped intermediate calculations. There are used also the already available solutions (23) for $\mathbf{K_2}(e,0)$ and (21) for $\mathbf{K_1}(e,0)$. From paper [3] we apply the following formulas: (20) for $\mathbf{A_1}(e,0)$, (21) for $\mathbf{A_2}(e,0)$, (28) for $\mathbf{J_1}(e,0)$ and (35) for $\mathbf{J_2}(e,0)$, respectively. Further evaluations lead to the seeking for final analytical result for the integral $\mathbf{K_3}(e,\dot{e} = 0)$:

(25) $\mathbf{K_3}(e,0) = (\pi/2)(1 - e^2)^{-5/2}\mathbf{\{}(4 + 2e^2)ln\{2[2 - 3e^2 + e^4 - 2(1 - e^2)^{3/2}]e^{-2}[1 - (1 - e^2)^{1/2}]^{-1}\} - - 6 - e^2 + 6(1 - e^2)^{1/2}\mathbf{\}}.$

To evaluate analytically the remaining two integrals $\mathbf{K_4}(e,\dot{e} = 0)$ and $\mathbf{K_5}(e,\dot{e} = 0)$, we proceed in a very similar way. Here we only sketch these calculations.

(26)
$$\mathbf{K_4}(e,0) = [3(1 - e^2)]^{-1}\{5\mathbf{K_3}(e,0) - 2\mathbf{K_2}(e,0) - (1 + e^2)\mathbf{A_3}(e,0) + \mathbf{A_2}(e,0) + + (1 - e^2)[\mathbf{J_3}(e,0) - \mathbf{J_2}(e,0)]\} = = [3(1 - e^2)]^{-1}\{(\pi/2)(1 - e^2)]^{-5/2}[(20 + 10e^2)ln\mathbf{Z}(e,0) - 30 - 5e^2 + 30(1 - e^2)^{1/2}] - - 4\pi(1 - e^2)(1 - e^2)^{-5/2}ln\mathbf{Z}(e,0) - 4\pi(1 - e^2)(1 - e^2)^{-5/2}[- 1 + (1 - e^2)^{1/2}] - - \pi(2 + e^2)(1 + e^2)(1 - e^2)^{-5/2} + 2\pi(1 - e^2)(1 - e^2)^{-5/2} + + (1 - e^2)[\pi(2 + 3e^2)(1 - e^2)^{-7/2} - \pi(2 + e^2)(1 - e^2)(1 - e^2)^{-7/2}]\}.$$

As before, we have applied the already computed results: formulas (25) and (23) for $\mathbf{K_3}(e,\dot{e} = 0)$ and $\mathbf{K_2}(e,\dot{e} = 0)$, respectively. From paper [3] we have used the evaluations (21) for $\mathbf{A_2}(e,0)$, (22) for $\mathbf{A_3}(e,0)$, (35) for $\mathbf{J_2}(e,0)$ and, finally, (44) for $\mathbf{J_3}(e,0)$. Consequently, the simplification of the solution (26) may be expressed as follows:

(27)
$$\mathbf{K_4}(e,0) = (\pi/6)(1 - e^2)^{-7/2}[(12 + 18e^2)ln\mathbf{Z}(e,0) - 22 - 15e^2 + (22 + 8e^2)(1 - e^2)^{1/2}] = = (\pi/6)(1 - e^2)^{-7/2}\mathbf{\{}(12 + 18e^2)ln\{2[2 - 3e^2 + e^4 - 2(1 - e^2)^{3/2}]e^{-2}[1 - (1 - e^2)^{1/2}]^{-1}\} - - 22 - 15e^2 + (22 + 8e^2)(1 - e^2)^{1/2}\mathbf{\}}.$$

Correspondingly, the integral $\mathbf{K_5}(e,\dot{e} = 0)$ can be computed by means of $\mathbf{K_4}(e,0)$ (formula (27)), $\mathbf{K_3}(e,0)$ (formula (25)), $\mathbf{A_3}(e,0)$ (formula (22) from paper [3]), $\mathbf{A_4}(e,0)$ (formula (23) from paper [3]) and $\mathbf{J_4}(e,0)$ (formula (50) from paper [3]).

(28)
$$\mathbf{K_5}(e,0) = [4(1 - e^2)]^{-1}[7\mathbf{K_4}(e,0) - 3\mathbf{K_3}(e,0) + (1 - e^2)\mathbf{J_4}(e,0) - 2\mathbf{A_4}(e,0) + \mathbf{A_3}(e,0)] = = (\pi/24)(1 - e^2)^{-9/2}[(48 + 144e^2 + 18e^4)ln\mathbf{Z}(e,0) - 100 - 150e^2 - 9e^4 - (1/2)(12e^2 + 3e^4) + + (100 + 110e^2)(1 - e^2)^{1/2}].$$

Therefore, the final analytical solution for the integral $\mathbf{K_5}(e,0)$ is:

(29)
$$\mathbf{K_5}(e,0) = (\pi/48)(1 - e^2)^{-9/2}\mathbf{\{}(96 + 288e^2 + 36e^4)ln\{2[2 - 3e^2 + e^4 - - 2(1 - e^2)^{3/2}]e^{-2}[1 - (1 - e^2)^{1/2}]^{-1}\} - 200 - 312e^2 - 21e^4 + (200 + 220e^2)(1 - e^2)^{1/2}\mathbf{\}},$$

where we, of course, have used the short-notation definition (19) for the function $\mathbf{Z}(e,0)$. To conclude the matter, connected with the application of the analytical solutions of the integrals of the type $\mathbf{K_i}(e,\dot{e})$, ($\mathbf{i}$ = 1. …, 5), we shall write down in an explicit form some of these expressions (namely, for $\mathbf{i}$ = 3, 4 and 5), when the two-arguments pair $\{e, \dot{e}\}$ is replaced by $\{e - \dot{e}, - \dot{e}\}$. We do not give here the detailed computations, but the only the final results, including also the definition (20) for the function $\mathbf{Z}(e - \dot{e}, - \dot{e})$:

(30)
$$\mathbf{K_3}(e - \dot{e}, - \dot{e}) = \pi(2 + e^2)(1 - e^2)^{-5/2}ln\mathbf{\{}\{2 - 3e^2 + e^4 + 3e\dot{e} - 2e^3\dot{e} - \dot{e}^2 + e^2\dot{e}^2 + + (- 2 + 2e^2 - 3e\dot{e} + \dot{e}^2)(1 - e^2)^{1/2} + (- 2 + 2e^2 - e\dot{e})[1 - (e - \dot{e})^2]^{1/2} + + (2 - e^2 + e\dot{e})(1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2}\}e^{-2}\{1 - [1 - (e - \dot{e})^2]^{1/2}\}^{-1}\mathbf{\}} + + \pi(- e^2 + 2e^4 - e^6 + 4e\dot{e} - 6e^3\dot{e} + 2e^5\dot{e} - 3\dot{e}^2 + 4e^2\dot{e}^2 - e^4\dot{e}^2)\dot{e}^{-2}(1 - e^2)^{-5/2} + \pi(e^2 - 2e^4 + e^6 - 4e\dot{e} + + 7e^3\dot{e} - 3e^5\dot{e} - 9e^2\dot{e}^2 + 3e^4\dot{e}^2 + 4e\dot{e}^3 - e^3\dot{e}^3)\dot{e}^{-2}(1 - e^2)^{-2}[1 - (e - \dot{e})^2]^{-1/2} + 3\pi(1 - e^2)^{-2}.$$

(31) $\mathbf{K_4}(e - \dot{e}, - \dot{e}) = \pi(2 + 3e^2)(1 - e^2)^{-7/2}ln\mathbf{\{}\{2 - 3e^2 + e^4 + 3e\dot{e} - 2e^3\dot{e} - \dot{e}^2 + e^2\dot{e}^2 +$

$$+ (-2 + 2e^2 - 3e\dot{e} + \dot{e}^2)(1 - e^2)^{1/2} + (-2 + 2e^2 - e\dot{e})[1 - (e - \dot{e})^2]^{1/2} +$$
$$+ (2 - e^2 + e\dot{e})(1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2}\}e^{-2}\{1 - [1 - (e - \dot{e})^2]^{1/2}\}^{-1}\mathbf{\}} + (\pi/3)(2e^3 - 4e^5 + 2e^7 -$$
$$- 9e^2\dot{e} + 15e^4\dot{e} - 6e^6\dot{e} + 18e\dot{e}^2 - 15e^3\dot{e}^2 + 6e^5\dot{e}^2 - 13\dot{e}^3 + 4e^2\dot{e}^3 - 2e^4\dot{e}^3)\dot{e}^{-3}(1 - e^2)^{-5/2} +$$
$$+ (\pi/3)(-2e^3 + 6e^5 - 6e^7 + 2e^9 + 9e^2\dot{e} - 26e^4\dot{e} + 25e^6\dot{e} - 8e^8\dot{e} - 18e\dot{e}^2 + 43e^3\dot{e}^2 -$$
$$- 37e^5\dot{e}^2 + 12e^7\dot{e}^2 - 45e^2\dot{e}^3 + 23e^4\dot{e}^3 - 8e^6\dot{e}^3 + 18e\dot{e}^4 - 5e^3\dot{e}^4 + 2e^5\dot{e}^4)\dot{e}^{-3}(1 - e^2)^{-3}[1 - (e - \dot{e})^2]^{-1/2} +$$
$$+ (\pi/3)(11 + 4e^2)(1 - e^2)^{-3} + (2\pi/3)(1 - e^2)^{-5/2}.$$

(32) $$\mathbf{K_5}(e - \dot{e}, - \dot{e}) = (\pi/4)(8 + 24e^2 + 3e^4)(1 - e^2)^{-9/2} ln\mathbf{\{}\{2 - 3e^2 + e^4 + 3e\dot{e} - 2e^3\dot{e} - \dot{e}^2 + e^2\dot{e}^2 +$$
$$+ (-2 + 2e^2 - 3e\dot{e} + \dot{e}^2)(1 - e^2)^{1/2} + (-2 + 2e^2 - e\dot{e})[1 - (e - \dot{e})^2]^{1/2} +$$
$$+ (2 - e^2 + e\dot{e})(1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2}\}e^{-2}\{1 - [1 - (e - \dot{e})^2]^{1/2}\}^{-1}\mathbf{\}} +$$
$$+ (\pi/6)(-3e^4 + 9e^6 - 9e^8 + 3e^{10} + 16e^3\dot{e} - 44e^5\dot{e} + 40e^7\dot{e} - 12e^9\dot{e} - 36e^2\dot{e}^2 + 81e^4\dot{e}^2 - 63e^6\dot{e}^2 +$$
$$+ 18e^8\dot{e}^2 + 48e\dot{e}^3 - 48e^3\dot{e}^3 + 42e^5\dot{e}^3 - 12e^7\dot{e}^3 - 25\dot{e}^4 + 2e^2\dot{e}^4 - 10e^4\dot{e}^4 + 3e^6\dot{e}^4)\dot{e}^{-4}(1 - e^2)^{-7/2} +$$
$$+ (\pi/12)(6e^4 - 24e^6 + 36e^8 - 24e^{10} + 6e^{12} - 32e^3\dot{e} + 126e^5\dot{e} - 186e^7\dot{e} + 122e^9\dot{e} - 30e^{11}\dot{e} + 72e^2\dot{e}^2 -$$
$$- 269e^4\dot{e}^2 + 382e^6\dot{e}^2 - 245e^8\dot{e}^2 + 60e^{10}\dot{e}^2 - 96e\dot{e}^3 + 280e^3\dot{e}^3 - 367e^5\dot{e}^3 + 243e^7\dot{e}^3 - 60e^9\dot{e}^3 - 264e^2\dot{e}^4 +$$
$$+ 143e^4\dot{e}^4 - 119e^6\dot{e}^4 + 30e^8\dot{e}^4 + 96e\dot{e}^5 - 8e^3\dot{e}^5 + 23e^5\dot{e}^5 - 6e^7\dot{e}^5)\dot{e}^{-4}(1 - e^2)^{-4}[1 - (e - \dot{e})^2]^{-1/2} +$$
$$+ (5\pi/12)(10 + 11e^2)(1 - e^2)^{-4}.$$

6. Final analytical explicit evaluations for the integrals

$$\mathbf{L_i(e,\dot{e}) \equiv \int_0^{2\pi} [\mathit{ln}(1 + \mathit{e}cos\varphi)](1 + \mathit{e}cos\varphi)^{-1}[1 + (\mathit{e} - \dot{\mathit{e}})cos\varphi]^{-i}\, d\varphi\ , (i = 1, 2, 3)}$$

We have already obtained in the previous paper [4] the final analytical expressions for the integrals $\mathbf{L_0}(e)$, $\mathbf{K_0}(e)$ and $\mathbf{K_1}(e,\dot{e})$ (see formulas (3), (4) and (5), respectively, into the introduction of the present paper).We have also evaluated in an explicit form the integrals $\mathbf{K_i}(e,\dot{e})$, ($\mathbf{i}$ = = 2, …, 5), (see formulas (10), (13), (15) and (17), respectively). This circumstance enables us to apply the recurrence relations (formulas (13), (12) and (11) from paper [4]), which will be sufficient to write explicitly, as functions of $e(u)$ and $\dot{e}(u) \equiv de(u)/du$ the unknown functions $\mathbf{L_1}(e,\dot{e})$, $\mathbf{L_2}(e,\dot{e})$ and $\mathbf{L_3}(e,\dot{e})$. Strictly speaking, here we do not need to know the full analytical solutions of the integrals $\mathbf{K_4}(e,\dot{e})$ and $\mathbf{K_5}(e,\dot{e})$, because we interrupt the recurrence chain at the integral $\mathbf{L_3}(e,\dot{e})$, i.e., we need not to calculate for our purposes the integrals $\mathbf{L_i}(e,\dot{e})$ with $\mathbf{i} \geq 4$. $\mathbf{K_4}(e,\dot{e})$ and $\mathbf{K_5}(e,\dot{e})$ are evaluated for other reasons. Therefore, we can write, according to ((13), paper [4]), that:

(33) $$\mathbf{L_1}(e,\dot{e}) = (e/\dot{e})\mathbf{L_0}(e) - [(e - \dot{e})/\dot{e}]\mathbf{K_1}(e,\dot{e}) =$$
$$= -2\pi e\dot{e}^{-1}(1 - e^2)^{-1/2} ln\{[1 + (1 - e^2)^{1/2}][2(1 - e^2)]^{-1}\} - 2\pi(e - \dot{e})\dot{e}^{-1}[1 - (e - \dot{e})^2]^{-1/2}\times$$
$$\times ln\mathbf{\{}\{2 - 3e^2 + e^4 + 3e\dot{e} - 2e^3\dot{e} - \dot{e}^2 + e^2\dot{e}^2 + (-2 + 2e^2 - 3e\dot{e} + \dot{e}^2)(1 - e^2)^{1/2} +$$
$$+ (-2 + 2e^2 - e\dot{e})[1 - (e - \dot{e})^2]^{1/2} +$$
$$+ (2 - e^2 + e\dot{e})(1 - e^2)^{1/2}[1 - (e - \dot{e})^2]^{1/2}\}(e - \dot{e})^{-2}[1 - (1 - e^2)^{1/2}]^{-1}\mathbf{\}}.$$

In a fully analogous way, we are able to evaluate the other integrals $\mathbf{L_2}(e,\dot{e})$ and $\mathbf{L_3}(e,\dot{e})$, for which we are seeking for. After some simple but tedious algebra, without using the notation (18) for $\mathbf{Z}(e,\dot{e})$, we shall give the final analytical form for the solutions of $\mathbf{L_2}(e,\dot{e})$ and $\mathbf{L_3}(e,\dot{e})$. Taking into account the recurrence relation (12) from paper [4] and the solutions (10) for $\mathbf{K_2}(e,\dot{e})$ and (32) for $\mathbf{L_1}(e,\dot{e})$, we have:

(34)
$$\begin{aligned}&\mathbf{L_2}(e,\dot{e}) = (e/\dot{e})\mathbf{L_1}(e,\dot{e}) - [(e-\dot{e})/\dot{e}]\mathbf{K_2}(e,\dot{e}) = \\ &= -2\pi e^2\dot{e}^{-2}(1-e^2)^{-1/2}ln\{[1+(1-e^2)^{1/2}][2(1-e^2)]^{-1}\} - 2\pi(e^2 - e^4 + 3e^3\dot{e} - \dot{e}^2 - 3e^2\dot{e}^2 + e\dot{e}^3)\dot{e}^{-2}\times \\ &\times[1-(e-\dot{e})^2]^{-3/2}ln\mathbf{\{}\{2 - 3e^2 + e^4 + 3e\dot{e} - 2e^3\dot{e} - \dot{e}^2 + e^2\dot{e}^2 + (-2 + 2e^2 - 3e\dot{e} + \dot{e}^2)(1-e^2)^{1/2} + \\ &+ (-2 + 2e^2 - e\dot{e})[1-(e-\dot{e})^2]^{1/2} + (2 - e^2 + e\dot{e})(1-e^2)^{1/2}[1-(e-\dot{e})^2]^{1/2}\}(e-\dot{e})^{-2}\times \\ &\times[1-(1-e^2)^{1/2}]^{-1}\mathbf{\}} + 2\pi(e^2 - e\dot{e})\dot{e}^{-2}[1-(e-\dot{e})^2]^{-1/2} - 2\pi(e^2 - e^4 - 2e\dot{e} + 2e^3\dot{e} + \dot{e}^2 - e^2\dot{e}^2)\dot{e}^{-2}\times \\ &\times[1-(e-\dot{e})^2]^{-1}(1-e^2)^{-1/2} - 2\pi(e-\dot{e})\dot{e}^{-1}[1-(e-\dot{e})^2]^{-1}.\end{aligned}$$

Similarly, the recurrence formula (11) from paper [4], combined with the solutions (13) for $\mathbf{K_3}(e,\dot{e})$ and (34) for $\mathbf{L_2}(e,\dot{e})$, leads to the following result:

(35)
$$\begin{aligned}&\mathbf{L_3}(e,\dot{e}) = (e/\dot{e})\mathbf{L_2}(e,\dot{e}) - [(e-\dot{e})/\dot{e}]\mathbf{K_3}(e,\dot{e}) = \\ &= -2\pi e^3\dot{e}^{-3}(1-e^2)^{-1/2}ln\{[1+(1-e^2)^{1/2}][2(1-e^2)]^{-1}\} - \pi(2e^3 - 4e^5 + 2e^7 + 10e^4\dot{e} - 10e^6\dot{e} - \\ &- 5e^3\dot{e}^2 + 20e^5\dot{e}^2 - 2\dot{e}^3 - 5e^2\dot{e}^3 - 20e^4\dot{e}^3 + 5e\dot{e}^4 + 10e^3\dot{e}^4 - \dot{e}^5 - 2e^2\dot{e}^5)\dot{e}^{-3}[1-(e-\dot{e})^2]^{-5/2}\times \\ &\times ln\mathbf{\{}\{2 - 3e^2 + e^4 + 3e\dot{e} - 2e^3\dot{e} - \dot{e}^2 + e^2\dot{e}^2 + (-2 + 2e^2 - 3e\dot{e} + \dot{e}^2)(1-e^2)^{1/2} + (-2 + 2e^2 - e\dot{e})\times \\ &\times[1-(e-\dot{e})^2]^{1/2} + (2 - e^2 + e\dot{e})(1-e^2)^{1/2}[1-(e-\dot{e})^2]^{1/2}\}(e-\dot{e})^{-2}[1-(1-e^2)^{1/2}]^{-1}\mathbf{\}} - \\ &- \pi(3e^3 - 6e^5 + 3e^7 - 3e^2\dot{e} + 15e^4\dot{e} - 12e^6\dot{e} - 3e\dot{e}^2 - 15e^3\dot{e}^2 + 18e^5\dot{e}^2 + 3\dot{e}^3 + 9e^2\dot{e}^3 - 12e^4\dot{e}^3 - 3e\dot{e}^4 + \\ &+ 3e^3\dot{e}^4)\dot{e}^{-3}[1-(e-\dot{e})^2]^{-2}(1-e^2)^{-1/2} - \pi(2e^2 - 2e^4 + e\dot{e} + 6e^3\dot{e} - 3\dot{e}^2 - 6e^2\dot{e}^2 + 2e\dot{e}^3)\dot{e}^{-2}\times \\ &\times[1-(e-\dot{e})^2]^{-2} + \pi(3e^3 - 3e^5 - e^2\dot{e} + 9e^4\dot{e} - 2e\dot{e}^2 - 9e^3\dot{e}^2 + 3e^2\dot{e}^3)\dot{e}^{-3}[1-(e-\dot{e})^2]^{-3/2}.\end{aligned}$$

Consequently, the above solution (35) encloses the considered by us system of analytical solutions for the auxiliary integrals $\mathbf{L_i}(e,\dot{e})$, ($\mathbf{i}$ = 0, …, 3) (see definitions (1)) and $\mathbf{K_j}(e,\dot{e})$, ($\mathbf{j}$ = 1, …, 5) (see definitions (2)). As a rule, these definite integrals turn out to be complicate expressions of the assumed by us independent variables $e(u)$ and $\dot{e}(u) \equiv de(u)/du$. Nevertheless, we are enjoyed to establish the explicit form of the solutions. With a preliminary optimism, we postpone the problem of the simplification of the expressions, where these integrals will enter as auxiliary functions. It is important to note, that during the process of derivations, it becomes clear that the computed solutions are unique. That is to say, the application of the formulas will not lead to bifurcation problems, generated by the established solutions itself. Another good characteristic of the above considered solutions is that they passage continuously through some *suspected* peculiar points like $e(u) = 0$, $\dot{e}(u) = 0$, $e(u) - \dot{e}(u) = 0$, etc. This was discussed earlier many times, and the answer to the problem was favorable: these peculiar points do not cause troubles. Such kind of conclusions are essentially proved by the corresponding L'Hospital's rule for resolving of indeterminacies of the type 0/0. Therefore, the established expressions for $\mathbf{L_i}(e,\dot{e})$, ($\mathbf{i}$ = 0, …, 3) and $\mathbf{K_j}(e,\dot{e})$, ($\mathbf{j}$ = 1, …, 5) may be used without troubles about these singular points.

7. Conclusions

The present paper encloses an investigation associated with an analytical solving of several types of definite integrals. They are considered to be functions of the eccentricities $e(u)$ and their derivatives $\dot{e}(u) \equiv \equiv de(u)/du$ of the particle orbits, moving in the accretion discs with elliptical shapes. These integrals were not found solved in the existing mathematical handbooks and reference books in forms, which are appropriate for use, according to our aspiration to apply them in the theory of elliptical accretion flows. More concretely, the integrals, which we have considered both in the present investigation, and in the papers [3] and [4], are $\mathbf{L_i}(e,\dot{e})$, ($\mathbf{i}$ = 0, …, 3) (defined by formula (1)), $\mathbf{K_j}(e,\dot{e})$, ($\mathbf{j}$ = 1, …, 5) (defined by formula (2)), $\mathbf{A_i}(e,\dot{e})$, ($\mathbf{i}$ = 1, …, 5) (defined by formula (6)), $\mathbf{J_j}(e,\dot{e})$, ($\mathbf{j}$ = 1, …, 4) (defined by formula (7)) and $\mathbf{H_j}(e,\dot{e})$, ($\mathbf{j}$ = 1, …, 4) (defined by formula (8)). The situation in our case is that the analytical solutions are intended to be set into application for resolving of a **concrete** task. It is connected with the specific model of accretion discs [5], and introduces some limitations on the variables $e(u)$ and $\dot{e}(u)$, which are treated as independent ones. Aside from the circumstance that, by definition, $\dot{e}(u) \equiv de(u)/du$. The laters must obey three inequalities for all values of the independent variable $u \equiv ln(p)$, where p is the focal parameter of the particular particle elliptical orbit). Namely, (*i*) $|e(u)| < 1$, (*ii*) $|\dot{e}(u)| < 1$ and (*iii*) $|e(u) - \dot{e}(u)| < 1$. These restrictions arise, because the variable $e(u)$ is considered as an eccentricity and the *stationary* accretion flows in the model of Lyubarskij et al. [5] are *a priori*, by hypothesis, excluding any singular behavior of the accretion disc characteristics. This means that the phenomena like the propagation of shock waves are not taken into account. Therefore, the above mentioned constraints (*i*) – (*iii*) are, essentially imposed from physical reasons. Of course, to these must be added also the property that all physical characteristics in the model of Lyubarskij et al. [5] are described by means of *real* quantities. As a consequence, the integrands into the formulas (1), (2), (6), (7) and (8) include only *real* functions, and the corresponding integrals are also *real* functions on $e(u)$ and $\dot{e}(u)$. It has to be mentioned, that such a simplified situation may not occur for models other than [5]. For example, in the paper of Ogilvie [6] is introduced the notion *complex eccentricity* $\mathbf{E}(\mathbf{r})$ (where $\mathbf{r}$ is the radius-vector), in order to treat *the more general case*, when the particle orbits of the eccentric accretion discs *are not sharing a common longitude of periastron*. But we are not dealing with this, very probably, much more difficult for analytical solving problem. It is enough only to mention that in the model of Ogilvie [6], the dynamical equation, governing of the structure of the disc, is, generally speaking, no more ordinary differential equation, but particular one.

We conclude our remarks, stressing that *we have **not** performed* an analytical solving of the full mathematical problem, concerning the evaluation of the integrals
$\mathbf{L_i}(e,\dot{e})$, $\mathbf{K_j}(e,\dot{e})$, $\mathbf{A_i}(e,\dot{e})$, $\mathbf{J_j}(e,\dot{e})$, and $\mathbf{H_j}(e,\dot{e})$, (the indices **i** and **j** run the corresponding values, accepted by us, in the definitions (1), (2), (6), (7) and (8)). This is done for some particular cases, satisfying the above discussed restriction, imposed on the variables $e(u)$ and $\dot{e}(u) \equiv de(u)/du$ from physical reasonings. The so established expressions for these integrals (see also papers [3] and [4]) are presenting a complete system of solutions, which is sufficient for our purposes. It gives a possibility to investigate the behavior of some other integrals, which *directly* enter into the dynamical equation for the elliptical accretion discs, described by the model of Lyubarskij et al. [5]. For this reason, we have named the former five types of integrals "auxiliary integrals".

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АНАЛИТИЧНО ПРЕСМЯТАНЕ НА ДВА ИНТЕГРАЛА, ВЪЗНИКВАЩИ В ТЕОРИЯТА НА ЕЛИПТИЧНИТЕ АКРЕЦИОННИ ДИСКОВЕ. III. РЕШАВАНЕ НА ПЪЛНАТА СИСТЕМА ОТ СПОМАГАТЕЛНИ ИНТЕГРАЛИ, СЪДЪРЖАЩИ ЛОГАРИТМИЧНИ ФУНКЦИИ В ТЕХНИТЕ ИНТЕГРАНДИ

Д. Димитров

Резюме

Настоящето изследване затваря започнатите в по-ранните статии [3] и [4] аналитични оценки на някои видове определени интеграли. Тези решения са неоходими стъпки в посока на разкриването на математическата структура на динамичното уравнение, управляващо свойствата на *стационарните* елиптични акреционни дискове, чиито апсидни линии на **всички орбити** лежат върху една и съща линия [5]. Въпреки че разглежданата тук задача може да изглежда, на пръв поглед, като една чисто математическа такава, има някои ограничения от физическо естество върху променливите, влизащи като аргументи в интеграндите. В тази статия ние решаваме аналитично следните два определени интеграла, включващи в техните числители (като множител) логаритмичната функция $\boldsymbol{ln(1 + ecos\varphi)}$. Конкретно, ние намираме в явна форма решенията на интегралите

$\mathbf{L_i}(e,\dot{e}) \equiv \int_0^{2\pi}[ln(1 + ecos\varphi)](1 + ecos\varphi)^{-1}[1 + (e - \dot{e})cos\varphi]^{-i}\, d\varphi$, ($\mathbf{i}$ = 0, …, 3) и $\mathbf{K_j}(e,\dot{e}) \equiv$
$\equiv \int_0^{2\pi}[ln(1 + ecos\varphi)][1 + (e - \dot{e})cos\varphi]^{-j}\, d\varphi$, ($\mathbf{j}$ = 1, …, 5). Тук ние сме използвали

следните обозначения. φ е азимуталният ъгъл. Интегрирането по φ от 0 до 2π означава усредняване върху цялата траектория за всяка една частица от диска. Всяка такава частица се спуска по спирала към центъра на диска, движейки се по (квази-) елиптични орбити с фокални параметри $\boldsymbol{p}$. На тези параметри $\boldsymbol{p}$ е позволено да варират за различните елиптични орбити. В нашия подход на изчисляване, ние третираме $\boldsymbol{e(u)}$ и $\boldsymbol{\dot{e}(u)}$ като независими променливи. Физически наложените ограничения (които, до известна степен, водят до опростявания на задачите) са $|e(u)| < 1$, $|\dot{e}(u)| < 1$ и $|e(u) - \dot{e}(u)| < 1$ за всички допустими значения на $\boldsymbol{u}$. Тоест, между най-вътрешната и най-външната орбита на диска. Следователно, установените в тази статия аналитични решения за интегралите $\mathbf{L_i}(e,\dot{e})$, ($\mathbf{i}$ = 0, …, 3) и $\mathbf{K_j}(e,\dot{e})$, ($\mathbf{j}$ = 1, …, 5) са, вероятно, не най-общите такива даже в областта на реалния анализ. Въпреки това, те са достатъчни за нашата цел да се опрости динамичното уравнение.